

## *An Analysis of Highway On-ramp Merging by Queuing Theory*

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### Synopsis

In this paper we deal with traffic behaviours on a section of highway including an on-ramp by means of queuing theory. It is the purpose of this paper to provide the adequate capacity for highway on-ramps, which is useful for the design of on-ramps and the traffic control of highway.

The highway on-ramp merging is modeled as a queue and the system is solved. Then the maximum possible flows for merging from an on-ramp is obtained in a form of an function of through lane flows. The traffic capacity of an on-ramp is estimated from the relation between the average waiting time before merging and the incoming flow from an on-ramp, which is induced by the theory of queues.

### 1. Introduction

At a highway on-ramp junctions, the traffic disturbance caused by merging becomes a serious obstacle, so that sufficient considerations should be necessary for the design and the traffic control of highway on-ramps. Therefore the traffic behaviours at an on-ramp junction must be made clear and the traffic capacity must be adequately estimated.

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In this paper we approach to mergings of traffic from an on-ramp into an outer through lane of highway by use of queuing theory. The problems of highway merging have been studied by many researchers, and the queuing theory has been used by some authors (1,2,3,4,5). But most of them deal with static merging which means that a vehicle stopping on a minor road merges into a major road with the appearance of a minimum acceptable gap.

Contrary to the static merging, the word dynamic merging means that a vehicle incoming from the on-ramp drives on the acceleration lane and merges into the through lane with the appearance of a minimum acceptable gap. The minimum acceptable gap is about 5~6 sec. in case of static merging, but in case of dynamic merging it becomes about 1.5~2.0 sec.. Of course, the difference of speed between the through lane and the on-ramp influences the possibility of merging.

More important differences exist between the static merging and the dynamic merging. In static merging models the minimum acceptable gap is considered to be a service time, and the following vehicle stopping on a ramp should form a judgement first on the gap between the former vehicle and the following vehicle on the through lane. On the other hand in dynamic merging models, the gaps observed by a vehicle driving on the acceleration lane may be delayed from the gaps observed by the fore vehicle.

Accordingly the following vehicle may not always form a judgement for merging first on the gap to which the fore vehicle merged. If we suppose the arrivals of vehicles on the through lane are random, the time lags observed from vehicles on the acceleration lane are independent one another.

Moreover if two vehicles inflow to the same gap, there must be delays since the former vehicle starts to merge until the following vehicle starts to merge.

This paper deals with dynamic merging. As the service time both the acceptable gap and the time from the beginning to the end of merging are considered. By the application of queuing theory, the critical flows on a ramp such that the queue length on an acceleration lane does not increase infinitely is obtained.

The practical capacity at on-ramp junctions should be small than the critical flows. It is estimated from the viewpoint that the average waiting time or the average running distance on an acceleration lane does not exceed a fixed value.

## 2. Formulation of On-ramp Merging Model

Now we formulate a highway on-ramp merging model by use of queuing theory. We subject such an on-ramp merging as an on-ramp inflows into the outer through lane of highway, and assume the acceleration lane is long enough. At a merging section, lane changings between the inner lane and the outer lane are not considered and as the through lane flows only the outer lane flows is considered.

Now we suppose the arrivals of through lane traffic and on-ramp traffic subject to poisson distributions with average arrival rates  $\lambda$  and  $\nu$  respectively. Therefore the interval of arrivals which is expressed as a gap subjects to an exponential distribution with average value  $1/\lambda$  and  $1/\nu$  respectively.

The probability density function  $f(t_1)$  of a gap on the through lane and  $g(t_2)$  of a gap on the on-ramp are expressed by

$$f(t_1) = \lambda e^{-\lambda t_1}, \quad (1)$$

$$g(t_2) = \nu e^{-\nu t_2}. \quad (2)$$

It is assumed that the through lane traffic drives at a constant speed  $V_0$ , while the on-ramp traffic drives on the acceleration lane at a constant speed  $V_1$  until they start merging.

We suppose that a vehicle willing to merge forms a judgement of the possibility of merging when the fore vehicle finished merging. Then the following vehicle merges, if there exist time lags  $\tau_1$  between the fore vehicle and  $\tau_2$  between the rear vehicle on the through lane.

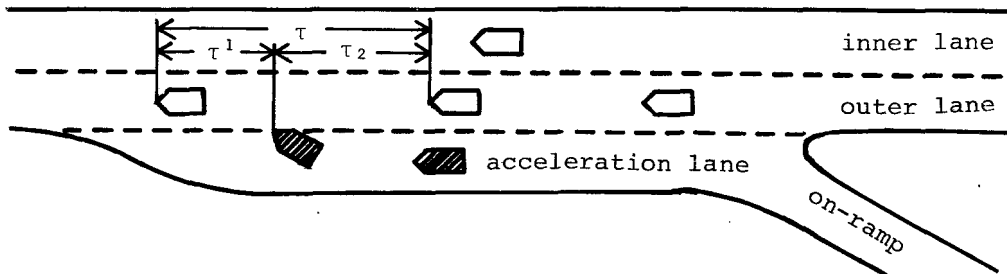


Fig.1 Minimum acceptable gap for merging.

If there exist no necessary time lags, they must wait until the necessary lags appear. This implies that the rule "first come first served" is assumed. We object the case that the merging is practiced in safety.

The time spent for merging is supposed to subject an exponential distribution with the average value  $1/\mu$ . Therefore the probability density function  $r(t_3)$  of merging time is given by

$$r(t_3) = \mu e^{-\mu t_3} . \quad (3)$$

Now let  $S_n(t)$  denote the probability that  $n$  vehicles are waiting for merging on the acceleration lane and the leading vehicle will merge with the appearance of the minimum acceptable gap  $\tau$ . In this situation, the minimum acceptable gap does not exist at time  $t$ . Accordingly the probability that the leading vehicle becomes to be able to merge in a very small interval of time  $dt$  is equal to the probability that a vehicle passes through a point  $V_0\tau_1$  ahead of the vehicle in  $dt$  under the condition the present lag is smaller than  $\tau$ , besides the next gap is larger than  $\tau$ . Therefore the probability is expressed by

$$\frac{1}{1-e^{-\lambda\tau}} \frac{\lambda(V_0-V_1)dt}{V_0} e^{-\lambda\tau} = \lambda(1-\psi) \frac{e^{-\lambda\tau}}{1-e^{-\lambda\tau}} dt ,$$

where,

$$\psi = V_1 / V_0 ,$$

$$\tau = \tau_1 + \tau_2 .$$

On the other hand, let  $D_n(t)$  denote the probability that there exist  $n$  vehicles on the acceleration lane at time  $t$  and the leading vehicle is just merging. Then the probability that the second vehicle becomes to be able to merge in a very small time interval  $dt$  is equal to the probability that the leading vehicle finishes the merging in a interval  $dt$  and the second vehicle have time lag  $\tau_1$  with the fore vehicle and  $\tau_2$  with the rear vehicle. Therefore the probability that the second vehicle becomes to be able to merge in a interval  $dt$  is equal to  $\lambda dt \exp(-\lambda\tau)$ . While if the second vehicle does not have time lag  $\tau_1$  or  $\tau_2$ , the vehicle becomes to wait for merging.

Moreover we denote the probability that there exist no vehicle on the acceleration lane at time  $t$  by  $R(t)$ . Then the next five equations should hold.

$$\begin{aligned} D_n(t+dt) = & D_{n-1}(t) v dt + D_n(t) (1-\mu dt - v dt) + D_{n+1}(t) \mu e^{-\lambda\tau} dt \\ & + S_n(t) \lambda (1-\psi) \frac{e^{-\lambda\tau}}{1-e^{-\lambda\tau}} dt , \quad (n \geq 2) , \end{aligned} \quad (4)$$

$$\begin{aligned} D_1(t+dt) = & D_1(t) (1-\mu dt - v dt) + D_2(t) \mu e^{-\lambda\tau} dt + S_1(t) \lambda (1-\psi) \frac{e^{-\lambda\tau}}{1-e^{-\lambda\tau}} dt \\ & + R(t) v dt e^{-\lambda\tau} , \quad (n \geq 2) , \end{aligned} \quad (5)$$

$$S_n(t+dt) = S_{n-1}(t)vdt + S_n(t)\{1-vdt-\lambda(1-\psi)\frac{e^{-\lambda\tau}}{1-e^{-\lambda\tau}}dt\} \\ + D_{n+1}(t)\mu(1-e^{-\lambda\tau})dt, \quad (n \geq 2), \quad (6)$$

$$S_1(t+dt) = S_1(t)\{1-vdt-\lambda(1-\psi)\frac{e^{-\lambda\tau}}{1-e^{-\lambda\tau}}dt\} \\ + D_2(t)\mu(1-e^{-\lambda\tau})dt + R(t)vdt(1-e^{-\lambda\tau}), \quad (7)$$

$$R(t+dt) = R(t)(1-vdt) + D_1(t)\mu dt. \quad (8)$$

If we rearrange the above equations and divide by  $dt$ , furthermore let  $dt$  approach zero, we have the following differential equations.

$$\frac{dD_n(t)}{dt} = vD_{n-1}(t) - (\mu+v)D_n(t) + \mu e^{-\lambda\tau}D_{n+1}(t) \\ + \lambda(1-\psi)\frac{e^{-\lambda\tau}}{1-e^{-\lambda\tau}}S_n(t), \quad (n \geq 2), \quad (9)$$

$$\frac{dD_1(t)}{dt} = -(\mu+v)D_1(t) + \mu e^{-\lambda\tau}D_2(t) \\ + \lambda(1-\psi)\frac{e^{-\lambda\tau}}{1-e^{-\lambda\tau}}S_1(t) + v e^{-\lambda\tau}R(t), \quad (10)$$

$$\frac{dS_n(t)}{dt} = vS_{n-1}(t) - \{v+\lambda(1-\psi)\frac{e^{-\lambda\tau}}{1-e^{-\lambda\tau}}\}S_n(t) \\ + \mu(1-e^{-\lambda\tau})D_{n+1}(t), \quad (n \geq 2), \quad (11)$$

$$\frac{dS_1(t)}{dt} = -\{v+\lambda(1-\psi)\frac{e^{-\lambda\tau}}{1-e^{-\lambda\tau}}\}S_1(t) + \mu(1-e^{-\lambda\tau})D_2(t) \\ + v(1-e^{-\lambda\tau})R(t), \quad (12)$$

$$\frac{dR(t)}{dt} = -vR(t) + \mu D_1(t). \quad (13)$$

Now we consider the limit state when  $t$  approaches infinity. In the limit state the system will be steady, i.e.  $dD_n(t)/dt=0$  ( $n \geq 1$ ),  $dS_n(t)/dt=0$  ( $n \geq 1$ ),  $dR(t)/dt=0$ . Besides  $D_n(t)$  ( $n \geq 1$ ),  $S_n(t)$  ( $n \geq 1$ ) and  $R(t)$  become to be independent of  $t$ , so that we can omit  $t$  from the above equations.

Thus the above differential equations are replaced by the difference equations of the form

$$vD_{n-1} - (\mu+v)D_n + \mu e^{-\lambda\tau}D_{n+1} + \lambda(1-\psi)\frac{e^{-\lambda\tau}}{1-e^{-\lambda\tau}}S_n = 0, \quad (n \geq 2), \quad (14)$$

$$-(\mu+v)D_1 + \mu e^{-\lambda\tau}D_2 + \lambda(1-\psi)\frac{e^{-\lambda\tau}}{1-e^{-\lambda\tau}}S_1 + v e^{-\lambda\tau}R = 0, \quad (15)$$

$$vS_{n-1} - \{v+\lambda(1-\psi)\frac{e^{-\lambda\tau}}{1-e^{-\lambda\tau}}S_n\} + \mu(1-e^{-\lambda\tau})D_{n+1} = 0, \quad (n \geq 2), \quad (16)$$

$$-\{v+\lambda(1-\psi)\frac{e^{-\lambda\tau}}{1-e^{-\lambda\tau}}\}S_1 + \mu(1-e^{-\lambda\tau})D_2 + v(1-e^{-\lambda\tau})R = 0, \quad (17)$$

$$-vR + \mu D_1 = 0. \quad (18)$$

From above equations, we have

$$vD_n - \mu D_{n+1} + vS_n = vD_{n-1} - \mu D_n + vS_{n-1} = \dots$$

$$\dots = vD_1 - \mu D_2 + vS_1 = vR - \mu D_1 = 0,$$

and this gives

$$S_n = \frac{\mu}{v} D_{n+1} - D_n, \quad (n \geq 1). \quad (19)$$

Substituting  $S_n$  into equation (14), we have

$$\begin{aligned} vD_{n-1} - \{\mu+v+\lambda(1-\psi)\frac{e^{-\lambda\tau}}{1-e^{-\lambda\tau}}\}D_n \\ + \{\mu e^{-\lambda\tau} + \frac{\mu\lambda}{v}(1-\psi)\frac{e^{-\lambda\tau}}{1-e^{-\lambda\tau}}\}D_{n+1} = 0, \quad (n \geq 2). \end{aligned}$$

This relation may be written by a recurrence of the form

$$AD_{n-1} - BD_n = c(AD_n - BD_{n+1}), \quad (n \geq 2), \quad (20)$$

where,

$$A = v, \quad (21)$$

$$B = \frac{1}{2} \left[ \mu+v+\lambda(1-\psi)\frac{e^{-\lambda\tau}}{1-e^{-\lambda\tau}} \pm \sqrt{\{\mu+v+\lambda(1-\psi)\frac{e^{-\lambda\tau}}{1-e^{-\lambda\tau}}\}^2 - 4v\{\mu e^{-\lambda\tau} + \frac{\mu\lambda}{v}(1-\psi)\frac{e^{-\lambda\tau}}{1-e^{-\lambda\tau}}\}} \right], \quad (22)$$

$$C = \frac{1}{2v} \left[ \mu+v+\lambda(1-\psi)\frac{e^{-\lambda\tau}}{1-e^{-\lambda\tau}} \mp \sqrt{\{\mu+v+\lambda(1-\psi)\frac{e^{-\lambda\tau}}{1-e^{-\lambda\tau}}\}^2 - 4v\{\mu e^{-\lambda\tau} + \frac{\mu\lambda}{v}(1-\psi)\frac{e^{-\lambda\tau}}{1-e^{-\lambda\tau}}\}} \right], \quad (23)$$

and  $A, B, C > 0$ .

We assume that

$$\frac{A}{B} < 1, \quad \frac{1}{C} < 1, \quad \frac{AC}{B} < 1,$$

then we have the solution of the recurrence formula (20),

$$D_n = \left(\frac{A}{B}\right)^{n-1} D_1 - \frac{K}{BC^{n-2}} \frac{1 - \left(\frac{AC}{B}\right)^{n-1}}{1 - \frac{AC}{B}}, \quad (n \geq 2). \quad (24)$$

where ,

$$K = AD_1 - BD_2.$$

From the assumption  $B > AC$ , the coefficients  $B, C$  must be

$$B = \frac{1}{2} \left( \mu + \nu + \lambda (1 - \psi) \frac{e^{-\lambda \tau}}{1 - e^{-\lambda \tau}} + \sqrt{\left\{ \mu + \nu + \lambda (1 - \psi) \frac{e^{-\lambda \tau}}{1 - e^{-\lambda \tau}} \right\}^2 - 4\nu \left\{ \mu e^{-\lambda \tau} + \frac{\mu \lambda}{\nu} (1 - \psi) \frac{e^{-\lambda \tau}}{1 - e^{-\lambda \tau}} \right\}} \right), \quad (25)$$

$$C = \frac{1}{2\nu} \left( \mu + \nu + \lambda (1 - \psi) \frac{e^{-\lambda \tau}}{1 - e^{-\lambda \tau}} - \sqrt{\left\{ \mu + \nu + \lambda (1 - \psi) \frac{e^{-\lambda \tau}}{1 - e^{-\lambda \tau}} \right\}^2 - 4\nu \left\{ \mu e^{-\lambda \tau} + \frac{\mu \lambda}{\nu} (1 - \psi) \frac{e^{-\lambda \tau}}{1 - e^{-\lambda \tau}} \right\}} \right). \quad (26)$$

From equations (15), (17), (18),  $D_1, D_2$  are written by

$$D_1 = \frac{\nu}{\mu} R,$$

$$D_2 = \frac{\nu R (B + CA - \mu e^{-\lambda \tau})}{\mu CB}.$$

Then the value of  $K$  is found by the condition

$$\sum_{n=1}^{\infty} D_n + \sum_{n=1}^{\infty} S_n + R = 1,$$

and this gives

$$K = \frac{\nu (E - B) (C - 1) (B - A)}{C\mu (CB - E)}, \quad (27)$$

$$R = \frac{(C - 1) (B - A)}{CB - E}, \quad (28)$$

where,

$$E = \mu e^{-\lambda \tau}. \quad (29)$$

Consequently we have the solution of the difference equations (14)~(18),

$$D_n = \frac{\nu}{\mu} \frac{(C-1)(B-A)}{(CB-E)(B-AC)} \left\{ (E-AC) \left(\frac{A}{B}\right)^{n-1} - (E-B) \left(\frac{1}{C}\right)^{n-1} \right\}, \quad (n \geq 1), \quad (30)$$

$$S_n = \frac{1}{\mu} \frac{(C-1)(B-A)(E-AC)(E-B)}{(CB-E)(B-AC)} \left\{ \left(\frac{A}{B}\right)^n - \left(\frac{1}{C}\right)^n \right\}, \quad (n \geq 1). \quad (31)$$

Now we seek the condition that the system gets to be steady. For this purpose, the inequalities

$$\frac{A}{B} < 1, \quad \frac{1}{C} < 1, \quad \frac{AC}{B} < 1,$$

must be hold. Since we decide B and C such that  $B > AC$ , the inequality  $A/B < 1$  holds if C satisfies  $1/C < 1$ . Hence the inequality

$$\frac{1}{2\nu} \left\{ \mu + \nu + \lambda(1-\psi) \frac{e^{-\lambda\tau}}{1-e^{-\lambda\tau}} - \sqrt{\left\{ \mu + \nu + \lambda(1-\psi) \frac{e^{-\lambda\tau}}{1-e^{-\lambda\tau}} \right\}^2 - 4\nu \left\{ \mu e^{-\lambda\tau} + \frac{\mu\lambda}{\nu}(1-\psi) \frac{e^{-\lambda\tau}}{1-e^{-\lambda\tau}} \right\}} \right\} > 1$$

must be hold. Rearranging the above inequality, we have

$$\nu < \frac{\mu\lambda(1-\psi)}{\mu(e^{-\lambda\tau} + e^{\lambda\tau} - 2) + \lambda(1-\psi)}. \quad (32)$$

We denote the right-hand side of the above inequality by  $\nu^*$ . The value of  $\nu^*$  gives the supremum of flow from the on-ramp such that vehicles of the acceleration lane do not increase infinitely. We consider the merging in the range under this critical flows from the on-ramp.

Now we seek the average value of queue length. The average number of vehicles on the acceleration lane including a merging vehicle is written by

$$\begin{aligned} L &= \sum_{n=1}^{\infty} n(S_n + D_n) \\ &= \frac{1}{\mu} \frac{(C-1)(B-A)}{(CB-E)(B-AC)} \left\{ \frac{ABE(E-AC)}{(B-A)^2} - \frac{CE(E-B)}{(C-1)^2} \right\}. \end{aligned} \quad (33)$$

In a general queue, the relation

$$L = \nu W$$

holds with respect to the average queue length  $L$ , the average waiting time  $W$  and the average arrival interval of time  $1/\nu$ . So that the average waiting time  $W$  a vehicle spends until the vehicle finishes the merging is written by



$$W = \frac{1}{v} L$$

$$= \frac{(C-1)(B-A)}{\mu v (CB-E)(B-AC)} \left\{ \frac{ABE(E-AC)}{(B-A)^2} - \frac{CE(E-B)}{(C-1)^2} \right\}, \quad (34)$$

and this gives the average running distance  $U$  on the acceleration lane until the vehicle finishes the merging

$$U = WV_1$$

$$= \frac{V_1 (C-1)(B-A)}{\mu v (CB-E)(B-AC)} \left\{ \frac{ABE(E-AC)}{(B-A)^2} - \frac{CE(E-B)}{(C-1)^2} \right\}. \quad (35)$$

### 3. Computational Results

Now we compute the values which are subtracted in the previous section, using parameters obtained by practical observations.

It is the minimum acceptable gap to influence severely to the possibility of merging.

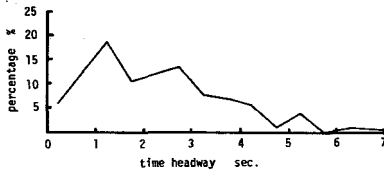


Fig. 2 Percentage of time headways between the merging vehicle and the fore vehicle.

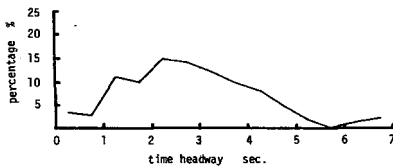


Fig. 3 Percentage of time headways between the merging vehicle and the rear vehicle.

Fig.2 shows the percentage of time headways between the merging vehicle and the fore vehicle on the through lane, and Fig.3 shows the percentage of time headways between the merging vehicle and the rear vehicle on the through lane (6).

In this Figure, some cases of forced merging are included. In general to practice a normal merging, time lags at least 0.7 sec. with the fore vehicle and 0.9 sec. with the rear vehicle are necessary. Therefore taking into account of some margin, we use  $\tau = 2.0$  sec. as the minimum acceptable gap.

Fig.4 shows the relation between the minimum necessary time gap and the number of vehicle

which merge to a same gap(7).

This proves that the merging is possible at the rate of one vehicle for 1.5 sec. Therefore we use  $1/\mu=1.5$  sec. as the average time spent for merging.

With respect to the speed, we assume that the relation between the average speed and the traffic density on the through lane is linear. Whence the speed  $V_0$  Km/h on the through lane is given as a function of flow  $Q$  v.p.h. by

$$V_0 = \frac{2400 + \sqrt{2400^2 - 2400 Q}}{60},$$

provided that the capacity is 2400 v.p.h. and the free speed is 80 Km/h at a section in which they are not influenced by merging. Further we assume the speed  $V_1$  of inflowing vehicles on the acceleration lane is 40 Km/h.

Using the above parameters we have the value of the critical flows from the on-ramp  $v^*$ , and this is shown in Fig.5.

We show the average number of vehicles on the acceleration lane  $W$  in Fig.6, and the average running distance on the acceleration lane  $U$  in Fig.7.

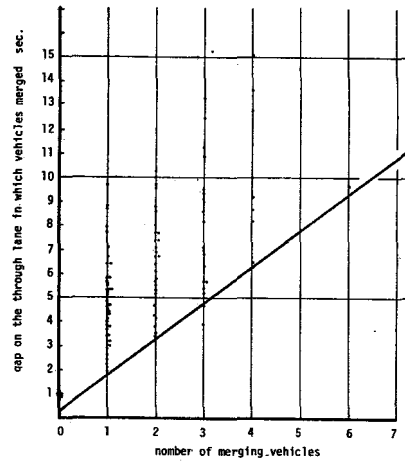


Fig.4 Time gap on the through lane and the number of merging vehicles.

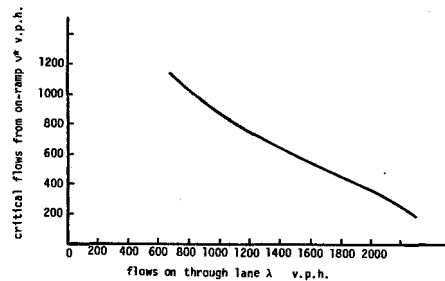


Fig.5 Critical flows from the on-ramp.

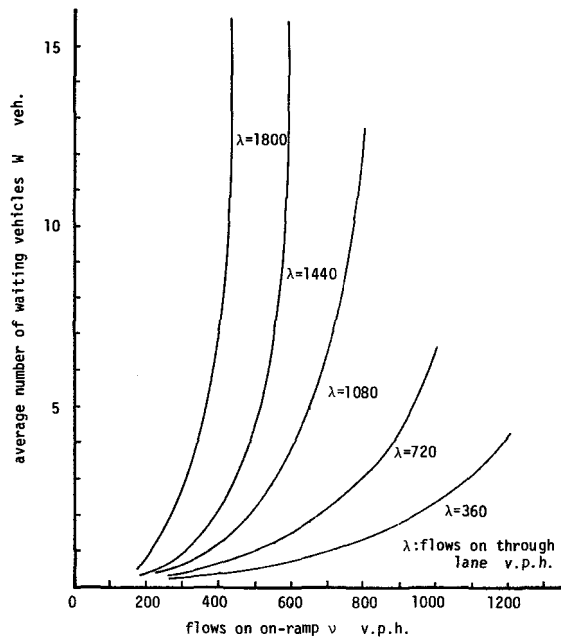


Fig. 6 Average number of vehicles on the acceleration lane.

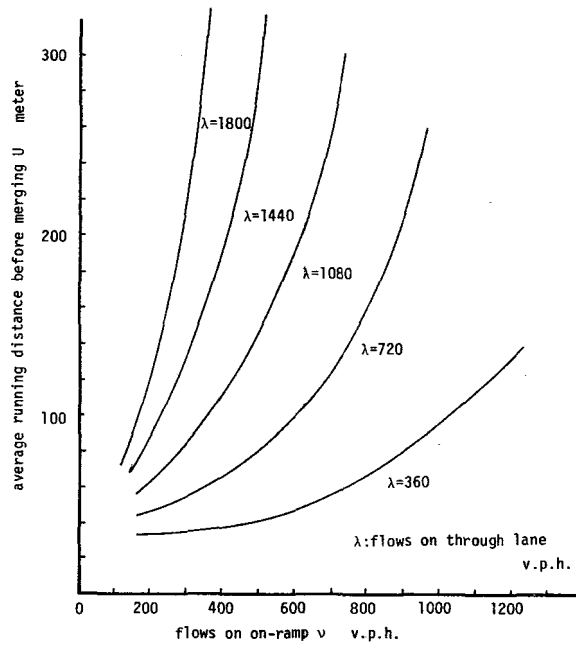


Fig. 7 Average running distance on the acceleration lane.

#### 4. Conclusions

The critical traffic volume from the on-ramp means the supremum of flow from the on-ramp such that vehicles on the acceleration lane do not increase infinitely. When the on-ramp flows approach to the critical values, the queue length will increase rapidly. Therefore we must estimate the traffic capacity at on-ramp junctions in the range the queue length is not so long. In practical, the length of acceleration lane will be a constraint, so that the acceptable queue length should be decided considering the length of the acceleration lane.

If an on-ramp junction with sufficient length of acceleration lane is objected and the average waiting time 20 sec. i.e. the average running distance of 222 meters is used as the criterion, then as the maximum acceptable flows from the on-ramp, we have 500 v.p.h. in case of middle traffic on through lane at about 1200 v.p.h. and 300 v.p.h. in case of heavy traffic at about 1800 v.p.h. Consequently the traffic capacity at an on-ramp junction is estimated at 1700 v.p.h. in case of middle traffic and 2100 v.p.h. in case of heavy traffic.

If we object an on-ramp junction with an acceleration lane about 200 meters length, we use the average waiting time of 10 sec. i.e. the average running distance of 111 meters as the criterion. Then the maximum acceptable flows from the on-ramp becomes 300 v.p.h. in case of middle traffic at about 1200 v.p.h. on the through lane and 200 v.p.h. in case of heavy traffic at about 1800 v.p.h.. Accordingly the traffic capacity at an on-ramp junction is estimated at 1500 v.p.h. in case of middle traffic and 2000 v.p.h. in case of heavy traffic.

These values agree approximately with the results investigated by means of the traffic simulation.

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References

- (1) T.C.Tanner: *Biometrika*, 49(1962), 163-170.
- (2) R.M.Oliver, E.F.Bisbee: *Operations Research*, 10(1962), 105-114.
- (3) D.H.Evans, R.Herman and G.H.Weiss, *Operations Research*, 12(1964), 832-857.
- (4) W.S.Jewell: *Operations Research*, 12(1964), 858-869.
- (5) D.E.Cleveland and D.G.Capell: H.R.B., Special Report 79 (1964), 51-96.
- (6) The Association of Highway Investigation: Traffic Behaviours of Merging and Branching, (1966), 16-45.
- (7) The same as the above, 56-105.